BENDING ANALYSIS OF LAMINATED COMPOSITE PLATES

A. Nosier* and F. F. Rajabzadeh
Department of Mechanical Engineering, Sharif University of Technology,
P.O.Box 11365-9567, Azadi Ave., Tehran, Iran

Abstract An idea is introduced to develop analytical solutions for any arbitrarily laminated composite plate subjected to transverse loads. The generality of the idea permits us to develop solutions for any combination of boundary conditions. In order to validate the accuracy of present method, numerical results are generated and compared with the Levy-type solutions of cross-ply laminates with various admissible boundary conditions. Excellent agreement is seen to exist between the present method and the exact Levy-type method.

Keywords: Laminated composite plates, Bending, Levy-type solution

INTRODUCTION

It is well known that for laminated composite plates a Levy-type solution exists only for cross-ply and anti-symmetric angle-ply laminates. Numerous investigators have used the Levy method to solve the governing equations of various equivalent single-layer plate theories for cross-ply and anti-symmetric angle-ply laminates [3-10]. It is the intention of the present study to introduce an idea which can be used to develop analytical solutions for any arbitrarily laminated composite plate. Furthermore the generality of the idea permits us to develop more general solution than Levy-type solution since in the latter one at least two opposite edges of the laminate must have simple supports. This restriction will be eliminated in the solution methodology that will be introduced in the present paper.

THEORETICAL FORMULATION

We are interested here to develop analytical solutions for any arbitrarily laminated composite plate. The laminated plate is composed of N orthotropic layers. The thickness of each ply is denoted by t_i and the plies have arbitrary fibre direction relative to the x-axis.The geometry of the plate is shown in Fig.1.Here,in order to introduce the idea ,We use a first-order shear deformation plate theory , although the method is general and can be used within any shear deformation plate and shell theory.

Displacement field and strains

Within the first-order shear deformation plate theory we consider the following displacement field:

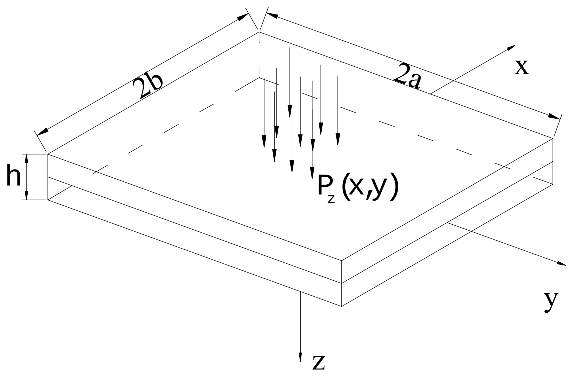

Fig.1 The geometry of laminated plate, coordinate system, and loading conditions

$$
\begin{aligned}
u(x,y,z) &= u_i(x)\bar{u}_j(y) + z\Psi_i(x)\bar{\Psi}_j(y) \\
v(x,y,z) &= v_i(x)\bar{v}_j(y) + z\Phi_i(x)\bar{\Phi}_j(y) \qquad (1) \\
w(x,y) &= w_i(x)\bar{w}_j(y)
\end{aligned}
$$

where $u_i(x)\bar{u}_j(y)$, $v_i(x)\bar{v}_j(y)$, $w_i(x)\bar{w}_j(y)$, denote the displacements of a point on the midplane of the laminate along x-,y- and z-coordinates,respectively, and $\Psi_i(x)\bar{\Psi}_j(y)$ and $\Phi_i(x)\bar{\Phi}_j(y)$ are the rotations of a transverse normal about the y- and x- axes, respectively. Upon substitution of Eqs.(1) into the linear strain-displacement relations [Reddy,1997] we obtain:

$$
\begin{aligned}
&\varepsilon_x = \varepsilon_x^{\circ} + zk_x \quad ; \; \varepsilon_y = \varepsilon_y^{\circ} + zk_y \quad ; \; \varepsilon_z = 0 \\
&\gamma_{xy} = \gamma_{xy}^{\circ} + zk_{xy} \; ; \; \gamma_{yz} = \gamma_{yz}^{\circ} \quad ; \; \gamma_{xz} = \gamma_{xz}^{\circ} \qquad (2)
\end{aligned}
$$

where

$$
\begin{aligned}
&\varepsilon_x^{\circ} = u_i'\bar{u}_j \quad ; \varepsilon_y^{\circ} = v_i\bar{v}'_j \quad ; \quad \gamma_{xyy}^{\circ} = u_i\bar{u}'_j + v_i'\bar{v}_j \\
&k_x = \Psi_i'\bar{\Psi}_j \; ; k_y = \Phi_i\bar{\Phi}'_j \; ; k_{xy} = \Psi_i\bar{\Psi}'_j + \Phi_i'\bar{\Phi}_j \qquad (3)
\end{aligned}
$$

* E-mail: noiser@sharif.edu

$\gamma^{\circ}_{yz} = \Phi_i \overline{\Phi}_j + w_i \overline{w'}_j \quad ; \quad \gamma^{\circ}_{xz} = \Psi_i \overline{\Psi}_j + w'_i \overline{w}_j$

In Eqs.(3) a prime indicates an ordinary derivate with respect to corresponding coordinate.

Equations of equilibrium

Using the principle of minimum total potential energy[Fung,1965],two sets of equilibrium equations are obtained.If the functions $\overline{u}_j, \overline{v}_j, \overline{w}_j, \overline{\Psi}_j$, and $\overline{\Phi}_j$ are assumed to be known, the first set of equilibrium equations can be shown to be :

$$\begin{aligned}
&\delta u_i : -\frac{dN_x^{ij}}{dx} + N_{xy1}^{ij} = 0\\
&\delta v_i : N_y^{ij} - \frac{dN_{xy2}^{ij}}{dx} = 0\\
&\delta \Psi_i : -\frac{dM_x^{ij}}{dx} + M_{xy1}^{ij} + Q_{x1}^{ij} = 0\\
&\delta \Phi_i : M_y^{ij} - \frac{dM_{xy2}^{ij}}{dx} + Q_{y1}^{ij} = 0\\
&\delta w_i : Q_{y2}^{ij} - \frac{dQ_{x2}^{ij}}{dx} = P_{z1}^{j}(x)
\end{aligned} \tag{4}$$

where the generalized stress and moment resultants are defined as:

$$(N_x^{ij}, N_y^{ij}, N_{xy1}^{ij}, N_{xy2}^{ij}) = \int_{-b}^{b} (N_x \overline{u}_j, N_y \overline{v'}_j, N_{xy} \overline{u'}_j, N_{xy} \overline{v}_j) dy$$

$$(M_x^{ij}, M_y^{ij}, M_{xy1}^{ij}, M_{xy2}^{ij}) = \int_{-b}^{b} (M_x \overline{\Psi}_j, M_y \overline{\Phi'}_j, M_{xy} \overline{\Psi'}_j, M_{xy} \overline{\Phi}_j) dy$$

$$(Q_{y1}^{ij}, Q_{y2}^{ij}, Q_{x1}^{ij}, Q_{x2}^{ij}) = \int_{-b}^{b} (Q_y \overline{\Phi}_j, Q_y \overline{w'}_j, Q_x \overline{\Psi}_j, Q_x \overline{w}_j) dy \tag{5}$$

with

$$P_{z1}^{j}(x) = \int_{-b}^{b} P_z(x,y) \overline{w}_j(y) dy \tag{6}$$

In Eqs.(5) we have :

$$\begin{aligned}
(N_x, N_y, N_{xy}) &= \int_{-h/2}^{h/2} (\sigma_x, \sigma_y, \sigma_{xy}) dz\\
(M_x, M_y, M_{xy}) &= \int_{-h/2}^{h/2} (\sigma_x, \sigma_y, \sigma_{xy}) z dz\\
(Q_y, Q_x) &= \int_{-h/2}^{h/2} (\sigma_{yz}, \sigma_{xz}) dz
\end{aligned} \tag{7}$$

The boundary conditions corresponding to Eqs.(4) require the specification of

either	u_i	or	N_x^{ij}	
either	v_i	or	N_{xy2}^{ij}	
either	Ψ_i	or	M_x^{ij}	at x=±a (8)
either	Φ_i	or	M_{xy2}^{ij}	
either	w_i	or	Q_{x2}^{ij}	

If ,on the other hand, the functions u_i, v_i, w_i, Ψ_i ,and Φ_i are assumed to be known, then we obtain the second set of equilibrium equations :

$$\begin{aligned}
&\delta \overline{u}_j : \overline{N}_x^{ij} - \frac{d\overline{N}_{xy1}^{ij}}{dy} = 0\\
&\delta \overline{v}_j : -\frac{d\overline{N}_y^{ij}}{dy} + \overline{N}_{xy2}^{ij} = 0\\
&\delta \overline{\Psi}_j : \overline{M}_x^{ij} - \frac{d\overline{M}_{xy1}^{ij}}{dy} + \overline{Q}_{x1}^{ij} = 0\\
&\delta \overline{\Phi}_j : -\frac{d\overline{M}_y^{ij}}{dy} + \overline{M}_{xy2}^{ij} + \overline{Q}_{y1}^{ij} = 0\\
&\delta \overline{w}_j : -\frac{d\overline{Q}_{y2}^{ij}}{dy} + \overline{Q}_{x2}^{ij} = P_{z2}^{i}(y)
\end{aligned} \tag{9}$$

In Eqs.(9) the generalized stress and moment resultants are defined as:

$$(\overline{N}_x^{ij}, \overline{N}_y^{ij}, \overline{N}_{xy1}^{ij}, \overline{N}_{xy2}^{ij}) = \int_{-a}^{a} (N_x u'_i, N_y v_i, N_{xy} u_i, N_{xy} v'_i) dx$$

$$(\overline{M}_x^{ij}, \overline{M}_y^{ij}, \overline{M}_{xy1}^{ij}, \overline{M}_{xy2}^{ij}) = \int_{-a}^{a} (M_x \Psi'_i, M_y \Phi_i, M_{xy} \Psi_i, M_{xy} \Phi'_i) dx \tag{10}$$

$$(\overline{Q}_{y1}^{ij}, \overline{Q}_{y2}^{ij}, \overline{Q}_{x1}^{ij}, \overline{Q}_{x2}^{ij}) = \int_{-a}^{a} (Q_y \Phi_i, Q_y w_i, Q_x \Psi_i, Q_x w'_i) dx$$

with

$$P_{z2}^{i}(y) = \int_{-a}^{a} P_z(x,y) w_i dx \tag{11}$$

where N_x, N_y, N_{xy}, M_x, M_y, M_{xy}, Q_x, and Q_y are the same as in Eqs.(7).

The boundary conditions corresponding to Eqs.(9) require the specification of

either	$\overline{u}_j$	or	$\overline{N}_{xy1}^{ij}$	
either	$\overline{v}_j$	or	$\overline{N}_y^{ij}$	
either	$\overline{\Psi}_j$	or	$\overline{M}_{xy1}^{ij}$	at y=±b (12)
either	$\overline{\Phi}_j$	or	$\overline{M}_y^{ij}$	
either	$\overline{w}_j$	or	$\overline{Q}_{y2}^{ij}$	

Lamina constitutive equations

The linear plane stress constitutive relations for the kth orthotropic lamina with respect to the laminate coordinate axes(see Fig.1.)are given by [Herakovich, 1998]:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix}^k = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix}^k \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}^k$$

$$\begin{Bmatrix} \sigma_{yz} \\ \sigma_{xz} \end{Bmatrix}^k = \begin{bmatrix} \overline{C}_{44} & \overline{C}_{45} \\ \overline{C}_{45} & \overline{C}_{55} \end{bmatrix}^k \begin{Bmatrix} \gamma^{\circ}_{yz} \\ \gamma^{\circ}_{xz} \end{Bmatrix}^k \tag{13}$$

where $[\overline{Q}_{ij}]$ is the transformed reduced stiffness matrix and $\overline{c}_{\alpha\beta}$ (α,β: 4,5) are elements of off-axis stiffness matrix of the kth layer.

Laminate constitutive equations

Upon substitution of Eqs.(2) into Eqs.(13) and the subsequent results into Eqs.(7) results:

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon^{\circ}_x \\ \varepsilon^{\circ}_y \\ \gamma^{\circ}_{xy} \\ k_x \\ k_y \\ k_{xy} \end{Bmatrix}$$

$$\begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = \begin{bmatrix} K^2 A_{44} & K^2 A_{45} \\ K^2 A_{45} & K^2 A_{55} \end{bmatrix} \begin{Bmatrix} \gamma^{\circ}_{yz} \\ \gamma^{\circ}_{xz} \end{Bmatrix} \qquad (14)$$

where

$$(A_{ij}, B_{ij}, D_{ij}) = \int_{-h/2}^{h/2} \overline{Q}_{ij}^{(k)} (1, z, z^2)\, dz \quad i, j: 1,2,6$$

$$A_{ij} = \int_{-h/2}^{h/2} \overline{C}_{ij}^{(k)}\, dz \quad i, j: 4,5 \qquad (15)$$

with K^2 being the shear correction factor.

Next we substitute Eqs.(3) into Eqs.(14) and the subsequent results into Eqs.(5) and Eqs.(10) to obtain:

$$\begin{Bmatrix} \overline{N}^{ij}_x \\ \overline{N}^{ij}_y \\ \overline{N}^{ij}_{xy1} \\ \overline{N}^{ij}_{xy2} \\ \overline{M}^{ij}_x \\ \overline{M}^{ij}_y \\ \overline{M}^{ij}_{xy1} \\ \overline{M}^{ij}_{xy2} \end{Bmatrix} = \begin{bmatrix} \overline{A}^i_{11} & \overline{A}^i_{12} & \overline{A}^i_{13} & \overline{A}^i_{14} & \overline{A}^i_{15} & \overline{A}^i_{16} & \overline{A}^i_{17} & \overline{A}^i_{18} \\ \overline{A}^i_{12} & \overline{A}^i_{22} & \overline{A}^i_{23} & \overline{A}^i_{24} & \overline{A}^i_{25} & \overline{A}^i_{26} & \overline{A}^i_{27} & \overline{A}^i_{28} \\ \overline{A}^i_{13} & \overline{A}^i_{23} & \overline{A}^i_{33} & \overline{A}^i_{34} & \overline{A}^i_{35} & \overline{A}^i_{36} & \overline{A}^i_{37} & \overline{A}^i_{38} \\ \overline{A}^i_{14} & \overline{A}^i_{24} & \overline{A}^i_{34} & \overline{A}^i_{44} & \overline{A}^i_{45} & \overline{A}^i_{46} & \overline{A}^i_{47} & \overline{A}^i_{48} \\ \overline{A}^i_{15} & \overline{A}^i_{25} & \overline{A}^i_{35} & \overline{A}^i_{45} & \overline{A}^i_{55} & \overline{A}^i_{56} & \overline{A}^i_{57} & \overline{A}^i_{58} \\ \overline{A}^i_{16} & \overline{A}^i_{26} & \overline{A}^i_{36} & \overline{A}^i_{46} & \overline{A}^i_{56} & \overline{A}^i_{66} & \overline{A}^i_{67} & \overline{A}^i_{68} \\ \overline{A}^i_{17} & \overline{A}^i_{27} & \overline{A}^i_{37} & \overline{A}^i_{47} & \overline{A}^i_{57} & \overline{A}^i_{67} & \overline{A}^i_{77} & \overline{A}^i_{78} \\ \overline{A}^i_{18} & \overline{A}^i_{28} & \overline{A}^i_{38} & \overline{A}^i_{48} & \overline{A}^i_{58} & \overline{A}^i_{68} & \overline{A}^i_{78} & \overline{A}^i_{88} \end{bmatrix} \begin{Bmatrix} \overline{u}_j \\ \overline{v'}_j \\ \overline{u'}_j \\ \overline{v}_j \\ \overline{\Psi}_j \\ \overline{\Phi'}_j \\ \overline{\Psi'}_j \\ \overline{\Phi}_j \end{Bmatrix}; \begin{Bmatrix} \overline{Q}^{ij}_{y1} \\ \overline{Q}^{ij}_{y2} \\ \overline{Q}^{ij}_{x1} \\ \overline{Q}^{ij}_{x2} \end{Bmatrix} = \begin{bmatrix} \overline{B}^i_{11} & \overline{B}^i_{12} & \overline{B}^i_{13} & \overline{B}^i_{14} \\ \overline{B}^i_{12} & \overline{B}^i_{22} & \overline{B}^i_{23} & \overline{B}^i_{24} \\ \overline{B}^i_{13} & \overline{B}^i_{23} & \overline{B}^i_{33} & \overline{B}^i_{34} \\ \overline{B}^i_{14} & \overline{B}^i_{24} & \overline{B}^i_{34} & \overline{B}^i_{44} \end{bmatrix} \begin{Bmatrix} \overline{\Phi}_j \\ \overline{w'}_j \\ \overline{\Psi}_j \\ \overline{w}_j \end{Bmatrix}$$

$$\begin{Bmatrix} N^{ij}_x \\ N^{ij}_y \\ N^{ij}_{xy1} \\ N^{ij}_{xy2} \\ M^{ij}_x \\ M^{ij}_y \\ M^{ij}_{xy1} \\ M^{ij}_{xy2} \end{Bmatrix} = \begin{bmatrix} A^j_{11} & A^j_{12} & A^j_{13} & A^j_{14} & A^j_{15} & A^j_{16} & A^j_{17} & A^j_{18} \\ A^j_{12} & A^j_{22} & A^j_{23} & A^j_{24} & A^j_{25} & A^j_{26} & A^j_{27} & A^j_{28} \\ A^j_{13} & A^j_{23} & A^j_{33} & A^j_{34} & A^j_{35} & A^j_{36} & A^j_{37} & A^j_{38} \\ A^j_{14} & A^j_{24} & A^j_{34} & A^j_{44} & A^j_{45} & A^j_{46} & A^j_{47} & A^j_{48} \\ A^j_{15} & A^j_{25} & A^j_{35} & A^j_{45} & A^j_{55} & A^j_{56} & A^j_{57} & A^j_{58} \\ A^j_{16} & A^j_{26} & A^j_{36} & A^j_{46} & A^j_{56} & A^j_{66} & A^j_{67} & A^j_{68} \\ A^j_{17} & A^j_{27} & A^j_{37} & A^j_{47} & A^j_{57} & A^j_{67} & A^j_{77} & A^j_{78} \\ A^j_{18} & A^j_{28} & A^j_{38} & A^j_{48} & A^j_{58} & A^j_{68} & A^j_{78} & A^j_{88} \end{bmatrix} \begin{Bmatrix} u'_i \\ v_i \\ u_i \\ v'_i \\ \Psi'_i \\ \Phi_i \\ \Psi_i \\ \Phi'_i \end{Bmatrix}; \begin{Bmatrix} Q^{ij}_{y1} \\ Q^{ij}_{y2} \\ Q^{ij}_{x1} \\ Q^{ij}_{x2} \end{Bmatrix} = \begin{bmatrix} B^j_{11} & B^j_{12} & B^j_{13} & B^j_{14} \\ B^j_{12} & B^j_{22} & B^j_{23} & B^j_{24} \\ B^j_{13} & B^j_{23} & B^j_{33} & B^j_{34} \\ B^j_{14} & B^j_{24} & B^j_{34} & B^j_{44} \end{bmatrix} \begin{Bmatrix} \Phi_i \\ w_i \\ \Psi_i \\ w'_i \end{Bmatrix} \qquad (16)$$

where the coefficients $A^j_{mn}, \overline{A}^i_{mn}$ (m,n:1..8) and $B^j_{mn}, \overline{B}^i_{mn}$ (m,n:1..4) are given in the appendix.

Governing equations

Equations (16) are substituted into Eqs.(4) and Eqs.(8) to yield two sets of governing equilibrium equations:

$$\begin{aligned} \delta u_i : & -A^j_{11} u''_i + A^j_{33} u_i - A^j_{14} v''_i + (A^j_{34} - A^j_{12}) v'_i + A^j_{23} v_i - A^j_{15} \Psi''_i \\ & + (A^j_{35} - A^j_{17}) \Psi'_i + A^j_{37} \Psi_i - A^j_{18} \Phi''_i + (A^j_{38} - A^j_{16}) \Phi'_i \\ & + A^j_{36} \Phi_i = 0 \end{aligned} \qquad (17\text{-a})$$

$$\begin{aligned} \delta v_i : & -A^j_{14} u''_i + (A^j_{12} - A^j_{34}) u'_i + A^j_{23} u_i - A^j_{44} v''_i + A^j_{22} v_i \\ & - A^j_{45} \Psi''_i + (A^j_{25} - A^j_{47}) \Psi'_i + A^j_{27} \Psi_i - A^j_{48} \Phi''_i + (A^j_{28} - A^j_{46}) \Phi'_i \\ & + A^j_{26} \Phi_i = 0 \end{aligned} \qquad (17\text{-b})$$

$$\begin{aligned} \delta \Psi_i : & -A^j_{15} u''_i + (A^j_{17} - A^j_{35}) u'_i + A^j_{37} u_i - A^j_{45} v''_i \\ & + (A^j_{47} - A^j_{25}) v'_i + A^j_{27} v_i - A^j_{55} \Psi''_i + (A^j_{77} + B^j_{33}) \Psi_i \\ & - A^j_{58} \Phi''_i + (A^j_{78} - A^j_{56}) \Phi'_i + (A^j_{67} + B^j_{13}) \Phi_i + B^j_{34} w'_i \\ & + B^j_{23} w_i = 0 \end{aligned} \qquad (17\text{-c})$$

$$\begin{aligned} \delta \Phi_i : & -A^j_{18} u''_i + (A^j_{16} - A^j_{38}) u'_i + A^j_{36} u_i - A^j_{48} v''_i \\ & + (A^j_{46} - A^j_{28}) v'_i + A^j_{26} v_i - A^j_{58} \Psi''_i + (A^j_{56} - A^j_{78}) \Psi'_i \\ & + (A^j_{67} + B^j_{13}) \Psi_i - A^j_{88} \Phi''_i + (A^j_{66} + B^j_{11}) \Phi_i + B^j_{14} w'_i \\ & + B^j_{12} w_i = 0 \end{aligned} \qquad (17\text{-d})$$

$$\begin{aligned} \delta w_i : & -B^j_{34} \Psi'_i + B^j_{23} \Psi_i - B^j_{14} \Phi'_i + B^j_{12} \Phi_i - B^j_{44} w''_i \\ & + B^j_{22} w_i = P^j_{z1}(x) \end{aligned} \qquad (17\text{-e})$$

and

$$\begin{aligned} \delta \overline{u}_j : & -\overline{A}^i_{33} \overline{u''}_j + \overline{A}^i_{11} \overline{u}_j - \overline{A}^i_{23} \overline{v''}_j + (\overline{A}^i_{12} - \overline{A}^i_{34}) \overline{v'}_j \\ & + \overline{A}^i_{14} \overline{v}_j - \overline{A}^i_{37} \overline{\Psi''}_j + (\overline{A}^i_{17} - \overline{A}^i_{35}) \overline{\Psi'}_j + \overline{A}^i_{15} \overline{\Psi}_j \\ & - \overline{A}^i_{36} \overline{\Phi''}_j + (\overline{A}^i_{16} - \overline{A}^i_{38}) \overline{\Phi'}_j + \overline{A}^i_{18} \overline{\Phi}_j = 0 \end{aligned} \qquad (18\text{-a})$$

$$\begin{aligned} \delta \overline{v}_j : & -\overline{A}^i_{23} \overline{u''}_j + (\overline{A}^i_{34} - \overline{A}^i_{12}) \overline{u'}_j + \overline{A}^i_{14} \overline{u}_j - \overline{A}^i_{22} \overline{v''}_j \\ & + \overline{A}^i_{44} \overline{v}_j - \overline{A}^i_{27} \overline{\Psi''}_j + (\overline{A}^i_{47} - \overline{A}^i_{25}) \overline{\Psi'}_j + \overline{A}^i_{45} \overline{\Psi}_j \\ & - \overline{A}^i_{26} \overline{\Phi''}_j + (\overline{A}^i_{46} - \overline{A}^i_{28}) \overline{\Phi'}_j + \overline{A}^i_{48} \overline{\Phi}_j = 0 \end{aligned} \qquad (18\text{-b})$$

$$\delta\overline{\Psi}_j : -\overline{A}^i_{37}\overline{u''}_j + (\overline{A}^i_{35} - \overline{A}^i_{17})\overline{u'}_j + \overline{A}^i_{15}\overline{u}_j - \overline{A}^i_{27}\overline{v''}_j + (\overline{A}^i_{25}-\overline{A}^i_{47})\overline{v'}_j + \overline{A}^i_{45}\overline{v}_j - \overline{A}^i_{77}\overline{\Psi''}_j + (\overline{A}^i_{55}+\overline{B}^i_{33})\overline{\Psi}_j - \overline{A}^i_{67}\overline{\Phi''}_j + (\overline{A}^i_{56}-\overline{A}^i_{78})\overline{\Phi'}_j + (\overline{A}^i_{58}+\overline{B}^i_{13})\overline{\Phi}_j + \overline{B}^i_{23}\overline{w'}_j + \overline{B}^i_{34}\overline{w}_j = 0 \quad \text{(18-c)}$$

$$\delta\overline{\Phi}_j : -\overline{A}^i_{36}\overline{u''}_j + (\overline{A}^i_{38} - \overline{A}^i_{16})\overline{u'}_j + \overline{A}^i_{18}\overline{u}_j - \overline{A}^i_{26}\overline{v''}_j + (\overline{A}^i_{28}-\overline{A}^i_{46})\overline{v'}_j + \overline{A}^i_{48}\overline{v}_j - \overline{A}^i_{67}\overline{\Psi''}_j + (\overline{A}^i_{78}-\overline{A}^i_{56})\overline{\Psi'}_j + (\overline{A}^i_{58}+\overline{B}^i_{13})\overline{\Psi}_j - \overline{A}^i_{66}\overline{\Phi''}_j + (\overline{A}^i_{88}+\overline{B}^i_{11})\overline{\Phi}_j + \overline{B}^i_{12}\overline{w'}_j + \overline{B}^i_{14}\overline{w}_j = 0 \quad \text{(18-d)}$$

$$\delta\overline{w}_j : -\overline{B}^i_{23}\overline{\Psi'}_j + \overline{B}^i_{34}\overline{\Psi}_j - \overline{B}^i_{12}\overline{\Phi'}_j + \overline{B}^i_{14}\overline{\Phi}_j - \overline{B}^i_{22}\overline{w''}_j + \overline{B}^i_{44}\overline{w}_j = P^i_{z2}(y) \quad \text{(18-e)}$$

THE SOLUTION PROCEDURE

Here,the solution procedure is discussed. Towards this end,a plate subjected to a uniform transverse load on its top surface is considered.The boundary conditions of the plate on opposite edges are assumed to be the same. In order to solve the governing equilibrium equations , as a first approximation a solution is assumed in the form of polynomials which satisfy the associated boundary conditions in either x or y directions. In order to fix the idea, let us assume that $u_1(x)$, $v_1(x)$, $\Psi_1(x)$, $\Phi_1(x)$, and $w_1(x)$ are chosen so that the boundary conditions at x=a and x=-a are identically satisfied.Next by letting i=1(see the appendix) coefficients $\overline{A}^1_{mn}$ (m,n:1..8) and $\overline{B}^1_{mn}$ (m,n:1..4) are found. Since these coefficients are constant, Eqs(18) will be five linear ordinary differential equations with constant coefficients. Also $P^1_{z2}(y)$ is found from Eq.(11) which will also be a constant since the transverse load is assumed be constant.Now Eqs.(18) may be solved analytically for any boundary conditions at y=±b to yield the general solution of $\overline{u}_j(y), \overline{v}_j(y), \overline{\Psi}_j(y), \overline{\Phi}_j(y), and\, \overline{w}_j(y)$ (j≥1). Next we can substitute the general solution of $\overline{u}_j, \overline{v}_j, \overline{\Psi}_j, \overline{\Phi}_j, and\, \overline{w}_j$ into Eq.(6) (and the appropriate relations in the appendix) to find the general expressions for $P^j_{z1}(x)$ and coefficients A^j_{mn} (j≥1, m,n:1..8) and B^j_{mn} (j≥1, m,n:1..4) which, here, will be constant. Now we can go back and solve Eqs.(17) for any boundary conditions at x=±a and obtain the general solution for $u_i(x)$, $v_i(x)$, $\Psi_i(x)$, $\Phi_i(x)$, and $w_i(x)$ (i≥2).This procedure will be continued until the solution is converged.

NUMERICAL RESULTS

The solution procedure outlined in the previous section is applied to symmetric cross-ply (0/90/90/0) square plates subjected to a uniformly distributed load. In all calculations the transverse shear correction factor is assumed to be 5/6. Numerical results are presented for two different span-to-thickness ratios (i.e.,2a/h) 150 and 25. The mechanical properties of the layers are taken to be those of a T300/5208 graphite/epoxy lamina(see [Herakovich,1998]):

E_1=132 GPa , E_2=E_3=10.8 Gpa
G_{12}=G_{13}=5.65 Gpa , G_{23}=3.38 Gpa
ν_{12} =ν_{13} =0.24 , ν_{23} =0.59

where 1,2, and 3 indicate the on-axis material coordinates. In the numerical results the non-dimentionalized parameters are length x/2a , center deflection $\overline{w} = \frac{w(x,0,0)E_2h^3}{P_\circ(2a)^4} * 10^3$ and $\overline{\sigma}_x = \frac{\sigma(x,0,-h/2)h^2}{P_\circ(2a)^4}$ *10. Here , $P_\circ$ denotes the intensity of the applied uniform load.Since the deflection $\overline{w} = \frac{w(x,0,0)E_2h^3}{P_\circ(2a)^4} * 10^3$ and $\overline{\sigma}_x = \frac{\sigma(x,0,-h/2)h^2}{P_\circ(2a)^4}$ *10. Here , $P_\circ$ denotes the intensity of the applied uniform load.Since the numerical results are compared with the Levy-type

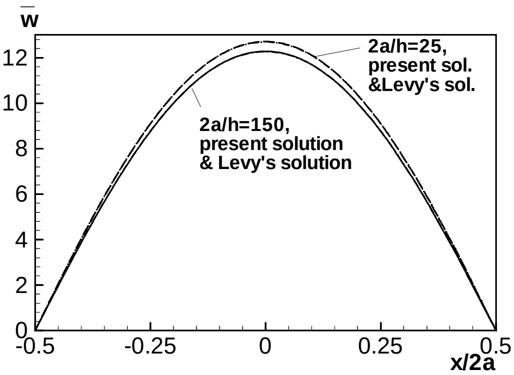

Fig.2 The nondimensionalized deflection for SS supports

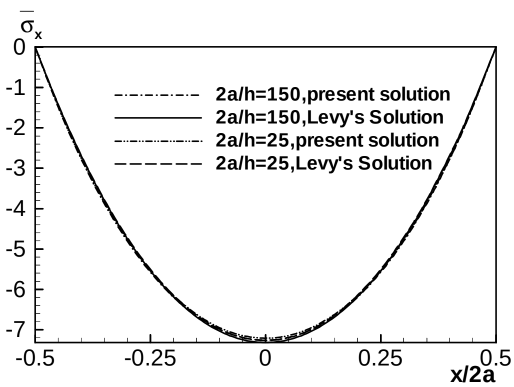

Fig.3 The distribution of the nondimensionalized normal stress $\overline{\sigma}_x$ for SS supports

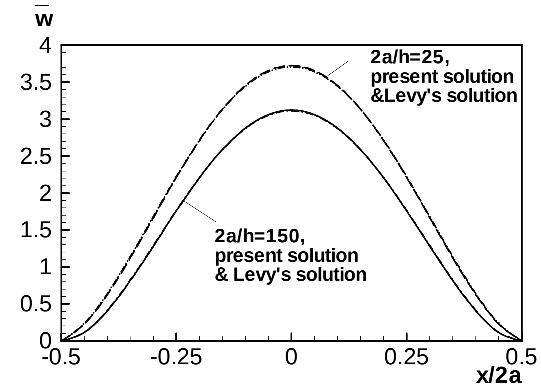

Fig.4 The nondimensionalized deflection for CC supports

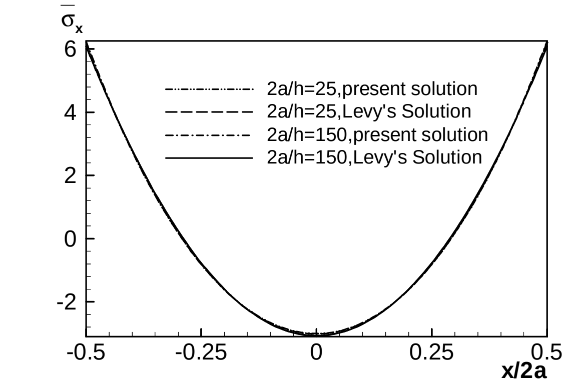

Fig.5 The distribution of nondimensionalized normal stress $\overline{\sigma}_x$ for CC supports

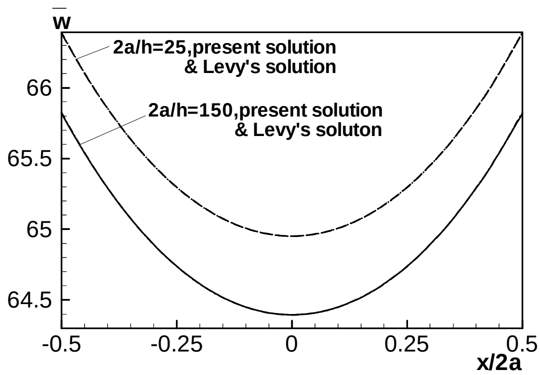

Fig.6 The nondimensionalized deflection for FF supports

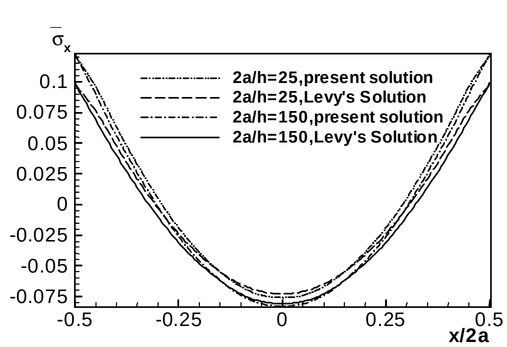

Fig. 7 The distribution of nondimensionalized normal stress $\overline{\sigma}_x$, for FF supports

solution, the edges at y=±b (see Fig.1.) will be considered invariably to be simply supported (S) while the remaining edges at x=±a, are assumed to have three different boundary supports : simply supported (S),clamped(C) and free(F).The designations SS,CC, and FF refer to the edge conditions at x=±a only.Figures 2 and 3 illustrate the variations of the nondimentionalized deflection $\overline{w}$ and normal stress $\overline{\sigma}_x$, respectively, when the edges at x=±a are simply supported(i.e.,SS). It is evident that a very good agreement exists between the Levy-type solution and the solution presented here.The variations of the deflection $\overline{w}$ and the normal stress $\overline{\sigma}_x$ when the edges at x=±a are clamped are shown in Figs.4 and 5, respectively.Again it is seen that excellent agreement exists between the present method and the exact Levy-type method. Finally Figs.6 and 7 present the variations of deflection $\overline{w}$ and the normal stress $\overline{\sigma}_x$ when the edges at x=±a are free(i.e.,FF). In the normal stress distribution shown in Fig.7 a maximum error of 23.5% exists at the free edges of the plate.It must be noted that for the solutions presented here, the trial and error procedure outlined in the previous section is continued until j=3 and i=4 which, on the other hand , results in a solution with an accuracy to the fourth decimal point for both $\overline{w}$ and $\overline{\sigma}_x$.

CONCLUSIONS

In this study analytical solutions based on an idea are developed for any arbitrarily laminated composite plate under various boundary conditions.Rectangular plates with simple supportes at two parallel edges and the remaining edges being either simply supported, clamped, or free are considered. The numerical results are compared with the Levy-type solutions of cross-ply laminates.Except for normal stress distributions at the free edges excellent agreements are seen to exist between the present method and the exact Levy-type method.

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APPENDIX

The coefficients $A_{ij}^{j}, B_{ij}^{j}, \bar{A}_{ij}^{i}, \bar{B}_{ij}^{i}$ are:

$$A_{11}^{j} = A_{11}\int_{-b}^{b}\bar{u}_j^2\,dy \qquad A_{12}^{j} = A_{12}\int_{-b}^{b}\bar{v}'_j\,\bar{u}_j\,dy$$

$$A_{13}^{j} = A_{16}\int_{-b}^{b}\bar{u}'_j\,\bar{u}_j\,dy \qquad A_{14}^{j} = A_{16}\int_{-b}^{b}\bar{v}_j\,\bar{u}_j\,dy$$

$$A_{15}^{j} = B_{11}\int_{-b}^{b}\bar{\Psi}_j\,\bar{u}_j\,dy \qquad A_{16}^{j} = B_{12}\int_{-b}^{b}\bar{\Phi}'_j\,\bar{u}_j\,dy$$

$$A_{17}^{j} = B_{16}\int_{-b}^{b}\bar{\Psi}'_j\,\bar{u}_j\,dy \qquad A_{18}^{j} = B_{16}\int_{-b}^{b}\bar{\Phi}_j\,\bar{u}_j\,dy$$

$$A_{22}^{j} = A_{22}\int_{-b}^{b}\bar{v}'^2_j\,dy \qquad A_{23}^{j} = A_{26}\int_{-b}^{b}\bar{u}'_j\,\bar{v}'_j\,dy$$

$$A_{24}^{j} = A_{26}\int_{-b}^{b}\bar{v}_j\,\bar{v}'_j\,dy \qquad A_{25}^{j} = B_{12}\int_{-b}^{b}\bar{\Psi}_j\,\bar{v}'_j\,dy$$

$$A_{26}^{j} = B_{22}\int_{-b}^{b}\bar{\Phi}'_j\,\bar{v}'_j\,dy \qquad A_{27}^{j} = B_{26}\int_{-b}^{b}\bar{\Psi}'_j\,\bar{v}'_j\,dy$$

$$A_{28}^{j} = B_{26}\int_{-b}^{b}\bar{\Phi}_j\,\bar{v}'_j\,dy \qquad A_{33}^{j} = A_{66}\int_{-b}^{b}\bar{u}'^2_j\,dy$$

$$A_{34}^{j} = A_{66}\int_{-b}^{b}\bar{v}_j\,\bar{u}'_j\,dy \qquad A_{35}^{j} = B_{16}\int_{-b}^{b}\bar{\Psi}_j\,\bar{u}'_j\,dy$$

$$A_{36}^{j} = B_{26}\int_{-b}^{b}\bar{\Phi}'_j\,\bar{u}'_j\,dy \qquad A_{37}^{j} = B_{66}\int_{-b}^{b}\bar{\Psi}'_j\,\bar{u}'_j\,dy$$

$$A_{38}^{j} = B_{66}\int_{-b}^{b}\bar{\Phi}_j\,\bar{u}'_j\,dy \qquad A_{44}^{j} = A_{66}\int_{-b}^{b}\bar{v}_j^2\,dy$$

$$A_{45}^{j} = B_{16}\int_{-b}^{b}\bar{\Psi}_j\,\bar{v}_j\,dy \qquad A_{46}^{j} = B_{26}\int_{-b}^{b}\bar{\Phi}'_j\,\bar{v}_j\,dy$$

$$A_{47}^{j} = B_{66}\int_{-b}^{b}\bar{\Psi}'_j\,\bar{v}_j\,dy \qquad A_{48}^{j} = B_{66}\int_{-b}^{b}\bar{\Phi}_j\,\bar{v}_j\,dy$$

$$A_{55}^{j} = D_{11}\int_{-b}^{b}\bar{\Psi}_j^2\,dy \qquad A_{56}^{j} = D_{12}\int_{-b}^{b}\bar{\Phi}'_j\,\bar{\Psi}_j\,dy$$

$$A_{57}^{j} = D_{16}\int_{-b}^{b}\bar{\Psi}'_j\,\bar{\Psi}_j\,dy \qquad A_{58}^{j} = D_{16}\int_{-b}^{b}\bar{\Phi}_j\,\bar{\Psi}_j\,dy$$

$$A_{66}^{j} = D_{22}\int_{-b}^{b}\bar{\Phi}'^2_j\,dy \qquad A_{67}^{j} = D_{26}\int_{-b}^{b}\bar{\Psi}'_j\,\bar{\Phi}'_j\,dy$$

$$A_{68}^{j} = D_{26}\int_{-b}^{b}\bar{\Phi}_j\,\bar{\Phi}'_j\,dy \qquad A_{77}^{j} = D_{66}\int_{-b}^{b}\bar{\Psi}'^2_j\,dy$$

$$A_{78}^{j} = D_{66}\int_{-b}^{b}\bar{\Phi}_j\,\bar{\Psi}'_j\,dy \qquad A_{88}^{j} = D_{66}\int_{-b}^{b}\bar{\Phi}_j^2\,dy$$

$$B_{11}^{j} = K^2A_{44}\int_{-b}^{b}\bar{\Phi}_j^2\,dy \qquad B_{12}^{j} = K^2A_{44}\int_{-b}^{b}\bar{w}'_j\,\bar{\Phi}_j\,dy$$

$$B_{13}^{j} = K^2A_{45}\int_{-b}^{b}\bar{\Psi}_j\,\bar{\Phi}_j\,dy \qquad B_{14}^{j} = K^2A_{45}\int_{-b}^{b}\bar{w}_j\,\bar{\Phi}_j\,dy$$

$$B_{22}^{j} = K^2A_{44}\int_{-b}^{b}\bar{w}'^2_j\,dy \qquad B_{23}^{j} = K^2A_{45}\int_{-b}^{b}\bar{\Psi}_j\,\bar{w}'_j\,dy$$

$$B_{24}^{j} = K^2A_{45}\int_{-b}^{b}\bar{w}_j\,\bar{w}'_j\,dy \qquad B_{33}^{j} = K^2A_{55}\int_{-b}^{b}\bar{\Psi}_j^2\,dy$$

$$B_{34}^{j} = K^2A_{55}\int_{-b}^{b}\bar{w}_j\,\bar{\Psi}_j\,dy \qquad B_{44}^{j} = K^2A_{55}\int_{-b}^{b}\bar{w}_j^2\,dy$$

$$\bar{A}_{11}^{i} = A_{11}\int_{-a}^{a}u'^2_i\,dx \qquad \bar{A}_{12}^{i} = A_{12}\int_{-a}^{a}v_iu'_i\,dx$$

$$\bar{A}_{13}^{i} = A_{16}\int_{-a}^{a}u_iu'_i\,dx \qquad \bar{A}_{14}^{i} = A_{16}\int_{-a}^{a}v'_iu'_i\,dx$$

$$\bar{A}_{15}^{i} = B_{11}\int_{-a}^{a}\Psi'_iu'_i\,dx \qquad \bar{A}_{16}^{i} = B_{12}\int_{-a}^{a}\Phi_iu'_i\,dx$$

$$\bar{A}_{17}^{i} = B_{16}\int_{-a}^{a}\Psi_iu'_i\,dx \qquad \bar{A}_{18}^{i} = B_{16}\int_{-a}^{a}\Phi'_iu'_i\,dx$$

$$\bar{A}_{22}^{i} = A_{22}\int_{-a}^{a}v_i^2\,dx \qquad \bar{A}_{23}^{i} = A_{26}\int_{-a}^{a}u_iv_i\,dx$$

$$\bar{A}_{24}^{i} = A_{26}\int_{-a}^{a}v'_iv_i\,dx \qquad \bar{A}_{25}^{i} = B_{12}\int_{-a}^{a}\Psi'_iv_i\,dx$$

$$\bar{A}_{26}^{i} = B_{22}\int_{-a}^{a}\Phi_iv_i\,dx \qquad \bar{A}_{27}^{i} = B_{26}\int_{-a}^{a}\Psi_iv_i\,dx$$

$$\bar{A}_{28}^{i} = B_{26}\int_{-a}^{a}\Phi'_iv_i\,dx \qquad \bar{A}_{33}^{i} = A_{66}\int_{-a}^{a}u_i^2\,dx$$

$$\bar{A}_{34}^{i} = A_{66}\int_{-a}^{a}v'_iu_i\,dx \qquad \bar{A}_{35}^{i} = B_{16}\int_{-a}^{a}\Psi'_iu_i\,dx$$

$$\bar{A}_{36}^{i} = B_{26}\int_{-a}^{a}\Phi_iu_i\,dx \qquad \bar{A}_{37}^{i} = B_{66}\int_{-a}^{a}\Psi_iu_i\,dx$$

$$\bar{A}_{38}^{i} = B_{66}\int_{-a}^{a}\Phi'_iu_i\,dx \qquad \bar{A}_{44}^{i} = A_{66}\int_{-a}^{a}v'^2_i\,dx$$

$$\bar{A}_{45}^{i} = B_{16}\int_{-a}^{a}\Psi'_iv'_i\,dx \qquad \bar{A}_{46}^{i} = B_{26}\int_{-a}^{a}\Phi_iv'_i\,dx$$

$$\bar{A}_{47}^{i} = B_{66}\int_{-a}^{a}\Psi_iv'_i\,dx \qquad \bar{A}_{48}^{i} = B_{66}\int_{-a}^{a}\Phi'_iv'_i\,dx$$

$$\bar{A}_{55}^{i} = D_{11}\int_{-a}^{a}\Psi'^2_i\,dx \qquad \bar{A}_{56}^{i} = D_{12}\int_{-a}^{a}\Phi_i\Psi'_i\,dx$$

$$\bar{A}_{57}^{i} = D_{16}\int_{-a}^{a}\Psi_i\Psi'_i\,dx \qquad \bar{A}_{58}^{i} = D_{16}\int_{-a}^{a}\Phi'_i\Psi'_i\,dx$$

$$\bar{A}_{66}^{i} = D_{22}\int_{-a}^{a}\Phi_i^2\,dx \qquad \bar{A}_{67}^{i} = D_{26}\int_{-a}^{a}\Psi_i\Phi_i\,dx$$

$$\bar{A}_{68}^{i} = D_{26}\int_{-a}^{a}\Phi'_i\Phi_i\,dx \qquad \bar{A}_{77}^{i} = D_{66}\int_{-a}^{a}\Psi_i^2\,dx$$

$$\bar{A}_{78}^{i} = D_{66}\int_{-a}^{a}\Phi'_i\Psi_i\,dx \qquad \bar{A}_{88}^{i} = D_{66}\int_{-a}^{a}\Phi'^2_i\,dx$$

$$\bar{B}_{11}^{i} = K^2A_{44}\int_{-a}^{a}\Phi_i^2\,dx \qquad \bar{B}_{12}^{i} = K^2A_{44}\int_{-a}^{a}w_i\Phi_i\,dx$$

$$\bar{B}_{13}^{i} = K^2A_{45}\int_{-a}^{a}\Psi_i\Phi_i\,dx \qquad \bar{B}_{14}^{i} = K^2A_{45}\int_{-a}^{a}w'_i\Phi_i\,dx$$

$$\bar{B}_{22}^{i} = K^2A_{44}\int_{-a}^{a}w_i^2\,dx \qquad \bar{B}_{23}^{i} = K^2A_{45}\int_{-a}^{a}\Psi_iw_i\,dx$$

$$\bar{B}_{24}^{i} = K^2A_{45}\int_{-a}^{a}w'_iw_i\,dx \qquad \bar{B}_{33}^{i} = K^2A_{55}\int_{-a}^{a}\Psi_i^2\,dx$$

$$\bar{B}_{34}^{i} = K^2A_{55}\int_{-a}^{a}w'_i\Psi_i\,dx \qquad \bar{B}_{44}^{i} = K^2A_{55}\int_{-a}^{a}w'^2_i\,dx$$